THE BRANCH LOCUS FOR ONE-DIMENSIONAL PISOT TILING SPACES

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ABSTRACT. If φ is a Pisot substitution of degree d, then the inflation and substitution homeomorphism Φ on the tiling space $\mathcal{T}_\Phi$ factors via geometric realization onto a d-dimensional solenoid. Under this realization, the collection of Φ-periodic asymptotic tilings corresponds to a finite set that projects onto the branch locus in a d-torus. We prove that if two such tiling spaces are homeomorphic, then the resulting branch loci are the same up to the action of certain affine maps on the torus.

1. INTRODUCTION

In this paper we introduce the branch locus, a new topological invariant for one-dimensional Pisot substitution tiling spaces. A substitution on n letters is a map from an alphabet $\mathcal{A} = \{1, 2, \ldots, n\}$ into $\mathcal{A}^*$, where $\mathcal{A}^*$ is the collection of finite and nonempty words from $\mathcal{A}$. The *abelianization matrix* of φ is defined as $A = (a_{ij})$, where $a_{ij} = $ number of occurrences of i in $\varphi(j)$. The substitution φ is *Pisot* provided the Perron-Frobenius eigenvalue λ of A is a Pisot-Vijayaraghavan number ($\lambda > 1$ and all algebraic conjugates of λ are strictly inside the unit circle). The *degree* of φ is the degree of the minimal polynomial of λ.

Associated with any substitution φ, there is a tiling space $\mathcal{T}_\varphi$ consisting of certain tilings of the real line; if $(\lambda_1, \ldots, \lambda_n)$ is a positive left eigenvector of A, then $\mathcal{T}_\varphi$ consists of all tilings of $\mathbb{R}$ by translates of the prototiles $P_i = [0, \lambda_i]$, $i = 1, \ldots, n$, with the property that the word spelled out by any finite patch of consecutive tiles in the tiling occurs as a factor of $\varphi^m(i)$ for some $i \in \mathcal{A}$ and $m \in \mathbb{N}$. The substitution φ is *primitive* if for some $m \in \mathbb{N}$, every entry of A^m is strictly positive, and *aperiodic* if no element of $\mathcal{T}_\varphi$ is periodic under translation. For the remainder of the paper, all substitutions will be assumed primitive and aperiodic.

Define the topology of $\mathcal{T}_\varphi$ by stipulating that two tilings are close provided a small translate of one is identical to the other in a large neighborhood of the origin. Under the assumption that φ is primitive and aperiodic, $\mathcal{T}_\varphi$ is a continuum (a compact, connected metric space). For such a φ, *inflation and substitution* is the homeomorphism $\Phi : \mathcal{T}_\varphi \to \mathcal{T}_\varphi$ that replaces each tile $t + P_i = [t, \lambda_i + t]$ of a tiling T in $\mathcal{T}_\varphi$ by the patch $[\lambda t, \lambda \lambda_i + \lambda t]$ tiled by translates of prototiles following the pattern of the word $\varphi(i)$. There is also a minimal and uniquely ergodic $\mathbb{R}$-action on $\mathcal{T}_\varphi$, called the *translation flow*, given by $T = \{T_i\}_{i=-\infty}^{\infty} \mapsto T - t := \{T_i - t\}_{i=-\infty}^{\infty}$, for $t \in \mathbb{R}$.

The topology of a substitution tiling space is of interest for a number of reasons. Physics provides one source of motivation. Suppose that T is a tiling in the substitution tiling space $\mathcal{T}_\varphi$. Placing an atom at the end of each tile creates a one-dimensional material which is called a *quasi-crystal* if its diffraction spectrum is pure point (the atoms must be 'weighted' according to the tile types they lie in). Bombieri

and Taylor ([BT]) proved that if φ is Pisot, such a material has a nontrivial discrete component in its spectrum. Whether such a material has pure point spectrum when φ is irreducible unimodular Pisot (i.e., φ is Pisot with degree(λ) $= d = n = |\mathcal{A}|$, and det(A) $= \pm 1$) remains an open question. Lee, Moody and Solomyak ([LMS]) have proved that the diffraction spectrum of the material is pure point if and only if the dynamical spectrum of the translation flow on $\mathcal{T}_\varphi$ is pure discrete. It follows from [BSw2] that if $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ are homeomorphic tiling spaces, then the tiling flow on $\mathcal{T}_\varphi$ is pure discrete if and only if the tiling flow on $\mathcal{T}_\psi$ is pure discrete. That is, the question of whether or not a one-dimensional material built from a substitution has pure point diffraction spectrum is a *topological* question about the corresponding tiling space.

Substitution tiling spaces also arise in the study of hyperbolic attractors. R.F. Williams ([Wi1]) proved that every hyperbolic one-dimensional attractor is topologically conjugate with the shift map on the inverse limit of an expanding endomorphism of a branched one-manifold, and, with minor restrictions on the map of the branched one-manifold, all such inverse limits can be realized as hyperbolic attractors. More recently, Anderson and Putnam ([AP]) proved that inflation and substitution on a one-dimensional substitution tiling space is conjugate with the shift on the inverse limit of an expanding endomorphism of a branched one-manifold. As a consequence, every orientable hyperbolic one-dimensional attractor is either a substitution tiling space, for which the underlying manifold is branched, or a classical solenoid, for which the underlying manifold is the circle. Modeling an attractor as a tiling space provides a much clearer view of its global topology than one gets from considering an inverse limit description: moving along an arc component in the attractor is simply translating a tiling, and the patterns of consecutive tiles determine the recurrence properties of the translates.

Although the "inverse limit on branched manifolds" description of tiling spaces will not play an explicit role in this paper, the intuitive content of our main result, and the rationale for the terminology "branch locus" that we introduce, has its origin in that description. There are, of course, no actual branch points in a tiling space: in the one-dimensional case, every point has a neighborhood that is homeomorphic with the product of an arc and a Cantor set. Nevertheless, an inverse limit description of the tiling space gives a sequence of approximating branched one-manifolds. In the limit, the ghost of the branches can be observed in the existence of asymptotic composants: two distinct tilings T, T' are *asymptotic* provided $d(T - t, T' - t) \to 0$ as $t \to \infty$ or as $t \to -\infty$. The arc components of asymptotic tilings are called *asymptotic composants*. Partially sewing up such asymptotic composants results in a space that does have branching: the new, branched, space corresponds to an inverse limit on a branched manifold with periodic branch points.

Although this "ghost branching" in $\mathcal{T}_\varphi$ seems to have no clear location, we will see that, in case the substitution is Pisot, the branching occurs in well defined relative geometrical patterns. The appropriate underlying geometry is that of the d-dimensional torus, where d is the degree of λ. Our main result is that if $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ are homeomorphic tiling spaces, then their branch loci, nonempty finite sets of points in the d-torus that we define in §2, are equal modulo the action of a certain collection of affine endomorphisms of the torus. Thus the branch locus becomes a topological invariant. We illustrate this by distinguishing pairs of tiling spaces that are otherwise difficult to tell apart (see §3).

The idea for considering "branching" in one-dimensional tiling spaces arose in discussions the first author had with Søren Eilers regarding the topological content of the K_0-group of the Matsumoto algebra associated with a substitution. Eilers, Restorff and Ruiz ([ERR]) have shown that this (ordered) group is also a complete invariant of the Matsumoto algebra and, consequently, the Matsumoto algebra is a topological invariant of the tiling space.

How is the Matsumoto algebra (equivalently, its K_0-group) reflected in the topology of the tiling space? In §5 we show that the branch locus provides a partial answer. We use the branch locus to define the "Pisot part of the augmented dimension group", an ordered group that is a flow equivalence invariant of the substitution and an ordered subgroup of the augmented cohomology group of the tiling space, which, in turn, is closely related to the Matsumoto K_0-group (see [CE1], [CE2] for a description of the Matsumoto K_0-group in the substitutive setting and [BSm] for an account of the relationship between the Matsumoto K_0-group of a substitutive system and the augmented cohomology of the associated tiling space).

2. Geometric Realization

For convenience, we will use the "strand space" model for the tiling space (see [BK] and [BBK]) which we recall now. Let φ be a Pisot substitution of degree d on n letters with abelianization A and Perron-Frobenius eigenvalue λ. There is an (unique) A-invariant decomposition $\mathbb{R}^n = V \oplus W$ such that V contains a right Perron-Frobenius eigenvector ω associated with λ and $\dim V = d$[1]. We can always choose rational bases for V and W. There is a further $A|_V$-invariant splitting $V = \mathbb{E}^u \oplus \mathbb{E}^s$ obtained by letting $\mathbb{E}^s$ be the space orthogonal to a left Perron-Frobenius eigenvector of $A|_V$ and $\mathbb{E}^u$ be the span of ω. Let $\text{pr}_V : \mathbb{R}^n \to V$, $\text{pr}_s : V \to \mathbb{E}^s$, and $\text{pr}_u : V \to \mathbb{E}^u$ denote the projections, resp., along W, $\mathbb{E}^u$ and $\mathbb{E}^s$, and let Γ denote the A-invariant lattice $\text{pr}_V \mathbb{Z}^n$.

If $e_i, i = 1, \ldots, n$, are the standard basis vectors in $\mathbb{R}^n$, define $v_i := \text{pr}_V(e_i)$, and let $\sigma_i := \{tv_i : 0 \le t \le 1\}$ denote the oriented segment representing v_i. Even if $\sigma_i = \sigma_j$ for some $i \ne j$, we wish to distinguish between these segments: we call σ_i a *(labeled) segment of type i*. An oriented broken line $\gamma = \{\sigma_{i_k} + x_k\}$, $x_k \in V$, consisting of a collection of translated copies of the basic segments meeting tip-to-tail and with connected union, will be called a *strand*. We denote the space of bi-infinite strands in V by

$$\mathcal{F} := \{\gamma : \gamma \text{ is a bi-infinite strand in } V\}.$$

The substitution φ induces the *inflation and substitution map* $\Phi : \mathcal{F} \to \mathcal{F}$ as follows: for each edge (translated segment) $\sigma_{i_k} + x_k$ in strand γ, replace that edge by the collection of edges $\sigma_{j_1} + Ax_k$, $\sigma_{j_2} + Ax_k + v_{j_1}, \ldots, \sigma_{j_\ell} + Ax_k + v_{j_1} + \cdots + v_{j_{\ell-1}}$, where $\varphi(i_k) = j_1 j_2 \cdots j_\ell$. That is, $\Phi(\gamma)$ is obtained by applying the linear map A to each edge of γ, then breaking up the image into translated segments following the pattern determined by applying φ to the type of the edge.

For $R > 0$, let $\mathcal{F}^R$ denote the subset of $\mathcal{F}$ consisting of those strands all of whose edges are within distance R of $\mathbb{E}^u$. There is then an R_0 so that

$$\mathcal{F}_\varphi := \bigcap_{n \ge 0} \Phi^n(\mathcal{F}^R)$$

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[1] V is the kernel of $m(A)$, if $m(x)$ denotes the minimal polynomial of λ.

is independent of $R \geq R_0$. The set $\mathcal{F}_\varphi$ has a natural metric topology in which γ and γ' are close if a small translate of γ by a vector in V lines up exactly with γ', segment types being considered, in a large neighborhood of the origin. With respect to this topology, Φ and the *translation flow*,

$$\gamma = \{\sigma_{i_k} + x_k\} \mapsto \gamma - t := \{\sigma_{i_k} + x_k - t\omega\},$$

are continuous.

In some cases there may be a few translation orbits in $\mathcal{F}_\varphi$ that correspond to strands with acceptable heads and tails but that are joined in an unnatural way (this happens, for instance, when there are letters a and b with $\varphi(a) = a \cdots$, $\varphi(b) = \cdots b$, but the word ba never occurs in $\varphi^m(i)$, $i \in \mathcal{A}$, $m \in \mathbb{N}$). We eliminate these chimeras by defining $\mathcal{T}_\varphi$ to be the ω-limit set, under translation flow, of any $\gamma \in \mathcal{F}_\varphi$. That is, for any $\gamma \in \mathcal{F}_\varphi$,

$$\mathcal{T}_\varphi := \bigcap_{T > 0} \mathrm{cl}\{\gamma - t : t \in [T, \infty)\}.$$

Inflation and substitution, $\Phi : \mathcal{T}_\varphi \to \mathcal{T}_\varphi$, is a homeomorphism from $\mathcal{T}_\varphi$ onto itself, and the translation flow on $\mathcal{T}_\varphi$ is minimal and uniquely ergodic.

One reason for using the rather elaborate strand space model of the tiling space is that it permits a simple and concrete definition of geometric realization. One could factor the tiling dynamics onto those of a solenoid, by choosing the lattice Γ in V and mapping $\gamma \in \mathcal{T}_\varphi$ to

$$(\min I_0 + \Gamma, \min I_1 + \Gamma, \ldots) \in \varprojlim F_{A|_V},$$

where $F_{A|_V} : V/\Gamma \to V/\Gamma$ is defined by $F_{A|_V}(v + \Gamma) = Av + \Gamma$, I_k is any edge in $\Phi^{-1}(\gamma)$ and $\min I_k$ is its initial vertex. This does give a well defined surjection of $\mathcal{T}_\varphi$ onto the d-dimensional solenoid $\varprojlim F_{A|_V}$ that semiconjugates inflation and substitution with the shift, as well as translation flow with a Kronecker action. However, to maximize the size of the factor, we will define a coarser, more natural lattice. Toward this end, denote the collection of *return vectors* by

$$\Theta(i) := \qquad \{v \in \Gamma : \text{ there exists } \gamma \in \mathcal{T}_\varphi \text{ containing edges } I, I',$$
$$\text{each of type } i, \text{ with } I' = I + v\}.$$

It is not difficult to show that the subgroup

$$\Sigma_\infty := \left\langle \bigcup_{k \in \mathbb{Z}} (A|_V)^k \Theta(i) \right\rangle$$

of V generated by $\bigcup_{k \in \mathbb{Z}} (A|_V)^k \Theta(i)$ is independent of $i \in \mathcal{A}$. The *return lattice*

$$\Sigma := \Sigma_\infty \cap \Gamma$$

is invariant under A and also of rank d.

Note: In case A is irreducible ($d = n$), then $\Sigma = \Gamma = \mathbb{Z}^n$. In §3, we will see an example where Σ is strictly coarser than Γ.

To define geometric realization onto a solenoid determined by Σ, we must determine some appropriate translations. We begin by noticing that for each $i, j \in \mathcal{A}$, there is a well defined "transition vector"

$w_{ij} \in \Gamma/\Sigma$ so that if $\gamma \in \mathcal{T}_\varphi$ and I and J are edges of γ of types i and j resp., then

$$(\min J - \min I) \bmod \Sigma = w_{ij}.$$

Then, since $w_{ij} + w_{jk} + w_{ki} = 0$ for all $i, j, k \in \mathcal{A}$, there are $u_i \in \Gamma/\Sigma$, for $i \in \mathcal{A}$, so that for all i, j,

$$w_{ij} = u_j - u_i.$$

We may then normalize the u_i so that

$$Au_i = u_{i'}$$

for all $i \in \mathcal{A}$, where i' denotes the initial letter of $\varphi(i)$. Letting $F_{A|_V}$ denote the map induced by A on the torus V/Σ, we have a well defined map $g_\varphi : \mathcal{T}_\varphi \to \varprojlim F_{A|_V}$ (we will always use the lattice Σ in place of Γ in the definition of $F_{A|_V}$) given by

$$g_\varphi(\gamma) = (\min I_0 + u_{i_0}, \min I_1 + u_{i_1}, \ldots)$$

where I_k is any edge of $\Phi^{-k}(\gamma)$ of type i_k. The map g_φ is called *geometric realization*.

Note: The map g_φ depends on the choice of the $u_i \in \Gamma/\Sigma$, which are not uniquely defined even after the normalization.

From results of [BK] and [BBK], the map g_φ is boundedly finite-to-one and almost everywhere m-to-1, where m is the *coincidence rank* of φ, and the tiling flow on $\mathcal{T}_\varphi$ has pure discrete spectrum if and only if $m = 1$. Moreover, g_φ is optimal in the sense that any other factoring of Φ onto a solenoidal shift factors through g_φ.[2]

Geometric realization expresses the underlying solenoidal nature of Pisot tiling spaces. But tiling spaces, unlike solenoids, are not homogeneous. *Asymptotic tilings* are one kind of inhomogeneity collapsed out by geometric realization. We say the tilings $\gamma, \gamma' \in \mathcal{T}_\varphi$ are *forward (backward, resp.) asymptotic* provided $d(\gamma - t, \gamma' - t) \to 0$ as $t \to \infty$ ($-\infty$, resp.).[3] If γ and γ' are asymptotic, where $\gamma \neq \gamma'$, there is a unique t so that $\gamma - t$ and $\gamma' - t$ are periodic under inflation and substitution. Moreover, the set $\mathcal{C}_\varphi$ consisting of those tilings $\gamma \in \mathcal{T}_\varphi$ for which there is a tiling $\gamma' \neq \gamma$ with γ and γ' asymptotic and Φ-periodic is finite and nonempty (see [BD1]). We will call the elements of $\mathcal{C}_\varphi$ *special tilings*.

Let $L : (V, \Sigma) \to (\mathbb{R}^d, \mathbb{Z}^d)$ be a linear isomorphism, and $F_L : V/\Sigma \to \mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ the induced isomorphism. Also, let $M = M_\varphi$ denote the integer matrix representing the linear isomorphism $L \circ A|_V \circ L^{-1}$ in the standard basis, and let $F_M : \mathbb{T}^d \hookleftarrow$ denote the corresponding toral endomorphism.

The *branch locus* of $\mathcal{T}_\varphi$ is the set

$$Br(\varphi) := (F_L \circ \pi_0 \circ g_\varphi)(\mathcal{C}_\varphi) \subset \mathbb{T}^d,$$

where $\pi_0 : \varprojlim F_{A|_V} \to V/\Sigma$ is projection onto the 0th coordinate. Note that $F_M(Br(\varphi)) = Br(\varphi)$. Also note that $Br(\varphi)$ depends not only on φ but also on the choice of $\{u_i\}$ in the definitions of g_φ and the linear isomorphism L made in the construction of g_φ. Our main theorem is that this dependence is limited.

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[2] For the translation flow on $\mathcal{T}_\varphi$, the Kronecker flow, $(z_0, z_1, \ldots) \mapsto (z_0 - t\omega, z_1 - t\lambda^{-1}\omega, \ldots)$ on $\varprojlim F_{A|_V}$ is the maximal equicontinuous factor.

[3] There is a weakening of asymptoticity, called regional proximality, with the property that $g_\varphi(\gamma) = g_\varphi(\gamma')$ if and only if γ and γ' are regionally proximal, see [Aus].

Theorem 1. *Suppose that* φ *and* ψ *are primitive, aperiodic, Pisot substitutions whose tiling spaces are homeomorphic. Then there are* $d \times d$ *integer matrices* S, T, *and* m_0, $m_1 \in \mathbb{N}$ *so that* $M_\varphi^{m_0} = ST$, $M_\psi^{m_1} = TS$, *and translations* τ_0, τ_1 *on* $\mathbb{T}^d$ *so that*

$$Br(\psi) = (\tau_0 \circ F_T)(Br(\varphi)) \quad and \quad Br(\varphi) = (\tau_1 \circ F_S)(Br(\psi))$$

In case λ is a *Pisot unit*, that is, $\det(A|_V) = \pm 1$; calculations are simplified.

Corollary 2. *Suppose that* φ *and* ψ *are primitive and aperiodic Pisot substitutions with* λ_φ *a Pisot unit of degree d, and that* $\mathcal{T}_\varphi$ *is homeomorphic with* $\mathcal{T}_\psi$. *Then there are a* $T \in GL(d, \mathbb{Z})$ *and a translation* τ *on* $\mathbb{T}^d$ *so that*

$$Br(\psi) = \tau \circ F_T(Br(\varphi)).$$

Before proving the theorem, we look at a few examples.

3. Examples

Example 1.

Consider the substitution $\varphi : \begin{cases} 1 & \to & 121 \\ 2 & \to & 312 \\ 3 & \to & 213 \end{cases}$. In this instance,

$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $\lambda = 3$ and $V = \mathbb{E}^u = \left\{ t \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\}$. We have $v_i = \mathrm{pr}_V e_i = \dfrac{1}{6} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, which we

denote by v, so that $\Gamma = \mathbb{Z}v$. Now consider the strand γ, fixed under Φ,

$$\gamma = \{\ldots, \sigma_1 - 2v, \sigma_2 - v, \sigma_1, \sigma_2 + v, \sigma_1 + 2v, \ldots\}.$$

That is, γ follows the pattern of the fixed word $\cdots 312.121 \cdots$ of φ. Clearly, if I and I' are two edges of γ of type 1, then $\min I' - \min I$ is of the form $2kv$, where $k \in \mathbb{Z}$ (and $k = 1$ occurs). Thus

$$\Sigma = \left\langle \bigcup_{k \in \mathbb{Z}} (A|_V)^k \Theta(1) \right\rangle \cap \Gamma = \left\langle \bigcup_{k \in \mathbb{Z}} (3^k)(2v) \right\rangle \cap \Gamma = 2\Gamma = 2\mathbb{Z}v.$$

The transition vectors are (mod Σ) $w_{21} = v$, $w_{31} = v$, and $w_{23} = 0$. Choosing $u_1 = 0$, $u_2 = v$, $u_3 = v$, we have $w_{ij} = u_j - u_i$ and $Au_i = 3u_i = u_{i'}$ (mod Σ), where $\varphi(i) = i' \cdots$. Finally, we map V/Σ onto $\mathbb{R}/\mathbb{Z} = \mathbb{T}^1$ by F_L, where $L : V \to \mathbb{R}$ is defined by $L(tv) = \frac{t}{2}v$. Then geometric realization of $\mathcal{T}_\varphi$ onto a 3-adic solenoid is given by $g_\varphi(\gamma) = (\min I_0 - u_{i_0}, \min I_1 - u_{i_1}, \ldots)$, where I_k is an edge of $\Phi^{-k}(\gamma)$ of type i_k and $\min I_k$ is taken mod Σ. The branch locus of φ is $(F_L \circ \pi_0 \circ g_\varphi)(C_\varphi)$.

There are procedures for finding asymptotic tilings described in [BD1] and [BDH], and an applet for this purpose can be found at [E1]. For this φ, there is a single pair of backward asymptotic tilings (of period 2 under Φ) and a single pair of forward asymptotic tilings (fixed by Φ) in $\mathcal{C}_\varphi = \{\alpha, \alpha', \beta, \beta'\}$, where

$$\alpha := \{\ldots, \sigma_2 - v, \sigma_1, \sigma_2 + v, \sigma_1 + 2v, \ldots\}$$

follows the pattern $\cdots 312.121 \cdots$,

$$\alpha' := \{\ldots, \sigma_3 - v, \sigma_1, \sigma_2 + v, \sigma_1 + 2v, \ldots\}$$

follows the pattern $\cdots 213.121 \cdots$,

$$\beta := \{\ldots, \sigma_1 - v, \sigma_3, \sigma_1 + v, \sigma_2 + 2v, \ldots\}$$

follows the pattern $\cdots 121.312 \cdots$ and

$$\beta' := \{\ldots, \sigma_1 - v, \sigma_2, \sigma_1 + v, \sigma_3 + 2v, \ldots\}$$

follows the pattern $\cdots 121.213 \cdots$. Thus $Br(\varphi) = \{0 + \mathbb{Z}, 1/2 + \mathbb{Z}\}$. The periodic forward asymptotic tilings are "halfway around" the tiling space from the periodic backward asymptotic tilings. $\qquad\square$

Note: One can show that $\Gamma/\Sigma \simeq \mathbb{Z}/h\mathbb{Z}$, where h is the *height* of the substitution φ (see [D]), if φ is constant length (as in Example 1) or, more generally, φ is Pisot of degree 1. In this sense, the group Γ/Σ is an extension of the notion of the height of a constant length substitution to the higher degree setting.

Example 2.

We will show that two particular tiling spaces are not homeomorphic by considering their branch loci. The Matsumoto K_0-group ([M]) and the crossing group ([BSw1] do not distinguish these spaces. Also, we have not been able to distinguish these spaces by the generalized Bowen-Franks trace ([BSw1]) or the proximality structures described in [BD2]. The (unordered) augmented cohomology groups of these spaces are isomorphic, but one can show that the Pisot parts of the augmented dimension groups (see §5) are not order isomorphic.

Consider the substitutions

$$\varphi : \begin{cases} a & \to aaa^{12}b^{16}aa \\ b & \to aba^{14}b^{14}ba \end{cases} \quad \text{and} \quad \psi : \begin{cases} a & \to aaaa^{11}b^{16}aa \\ b & \to aaba^{13}b^{14}ba \end{cases}.$$

Here $A_\varphi = A_\psi = \begin{pmatrix} 16 & 16 \\ 16 & 16 \end{pmatrix}$, $v_\varphi = v_\psi = \mathbb{E}^u = \left\{ t\begin{pmatrix} 1 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\}$, $v_1 = v_2 = v = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ for both φ and ψ, and $\Sigma_\varphi = \Gamma_\varphi = \Gamma_\psi = \Sigma_\psi = \mathbb{Z}v$. In geometric realization all the u_i can be taken to be 0. Choosing $L : tv \mapsto t$, one gets $Br(\varphi) = \{1/31 + \mathbb{Z}, 30/31 + \mathbb{Z}\}$, $Br(\psi) = \{2/31 + \mathbb{Z}, 30/31 + \mathbb{Z}\}$, and $M = M_\varphi = M_\psi = (32)$. So, if $\mathcal{T}_\varphi$ is homeomorphic with $\mathcal{T}_\psi$, there must be $S = (s)$ and $T = (t)$, where $s, t \in \mathbb{Z}$, an $m \in \mathbb{N}$ so that $M^m = ST$, and a translation τ on $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ so that $\tau \circ F_T Br(\varphi) = Br(\psi)$. The only possibility for t is $\pm 2^k$, where $k \in \mathbb{N}$. Then $\tau\left((\pm 2^k)\{1/31, 30/31\}\right) = \{2/31, 30/31\}(\text{mod}\,1)$ implies that $(\pm 2^k)(29/31) = (28/31)(\text{mod}\,1)$, hence that $(\pm 2^k)(29) = 28(\text{mod}\,31)$, or $\pm 2^{k+1} = 3(\text{mod}\,31)$, which is not the case for any $k \in \mathbb{N}$. It follows from Theorem 1 that $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ are not homeomorphic. $\qquad\square$

Example 3.

Define φ: $\begin{array}{ll} 1 & \to 1122111 \\ 2 & \to 1211111221 \end{array}$ and ψ: $\begin{array}{ll} 1 & \to 1121211 \\ 2 & \to 1211112211 \end{array}$.

The matrix $A = A_\varphi = A_\psi = \begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}$, so $\lambda = \lambda_\varphi = \lambda_\psi$ is a Pisot unit and Corollary 2 applies. As λ has degree 2, $\Sigma_\varphi = \Sigma_\psi = \mathbb{Z}^2$. Taking L to be the identity map, geometrical realization for both $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ takes the form $\gamma \mapsto \min I \pmod{\mathbb{Z}^2}$, where this last I is an edge of γ. Each of φ and ψ has a pair of backward and a pair of forward asymptotic tilings, fixed under inflation and substitution. Consider, for example, the tilings for φ following the patterns

$$\cdots 11221111\dot{1}22111 \cdots$$

and

$$\cdots 11221111\dot{2}11111221 \cdots ,$$

where the dot in each word above indicates the position of the origin. To locate the corresponding strands (call them γ_1 and γ_2), we seek $x \in \mathbb{R}^2$ so that

$$\Phi(\sigma_1 + x) = \{\sigma_1 + x - e_1, \sigma_1 + x, \sigma_2 + x + e_1, \ldots\}$$

or, equivalently,

$$\Phi(\sigma_2 + x) = \{\sigma_1 + x - e_1, \sigma_2 + x, \sigma_1 + x + e_2, \ldots\}.$$

Since

$$\Phi(\sigma_1 + x) = \{\sigma_1 + Ax, \sigma_1 + Ax + e_1, \ldots\}$$

we must have $Ax = x - e_1$. This yields $x = \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{pmatrix}$, a vertex of both γ_1 and γ_2. Thus

$$g_\varphi(\gamma_1) = g_\varphi(\gamma_2) = \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} \pmod{\mathbb{Z}^2} \in Br(\varphi).$$

Similar calculations yield:

$$Br(\varphi) = \left\{ \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} + \mathbb{Z}^2, \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} + \mathbb{Z}^2 \right\}$$

and

$$Br(\psi) = \left\{ \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} + \mathbb{Z}^2, \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} + \mathbb{Z}^2 \right\}.$$

Were $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ homeomorphic, there would be a $T \in GL(2, \mathbb{Z})$ and a translation τ on $\mathbb{T}^2$ so that $(\tau \circ F_T)(Br(\varphi)) = Br(\psi)$. In other words,

$$F_T\left[\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} - \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} + \mathbb{Z}^2 \right] = \pm \left[\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} + \mathbb{Z}^2 \right].$$

That is, $T \in GL(2, \mathbb{Z})$ would satisfy

$$\frac{1}{3} T \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \mathbb{Z}^2.$$

Since this is not possible, Corollary 2 implies that $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ are not homeomorphic. $\qquad\square$

Example 4.

$$\varphi := \begin{cases} a \to aaa^2b^4cac \\ b \to aba^7b^6c^5bbc \\ c \to aaba^6b^6c^3cbc \end{cases} \qquad \psi := \begin{cases} a \to aaab^4caac \\ b \to aaba^5b^5c^5bbbac \\ c \to aabbaa^4b^4c^3cbbac \end{cases}$$

with abelianization $A = \begin{pmatrix} 5 & 8 & 8 \\ 4 & 9 & 8 \\ 2 & 6 & 5 \end{pmatrix}$. The projection along $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ onto the Pisot subspace V is given by

$\mathrm{pr}_V = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}$. In this case the lattices Γ and Σ are equal, since return vectors span Γ over $\mathbb{Z}$ (the repeated letters aa, bb, and cc yield return vectors $v_i = \mathrm{pr}_V(e_i)$, $i = 1, 2, 3$).

The set $\{v_1, v_1 + v_2\} = \left\{ \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis for Σ over $\mathbb{Z}$. In these coordinates, $A|_V$ takes the form

$M = \begin{pmatrix} 5 & 16 \\ 4 & 13 \end{pmatrix}$.

There are one pair of fixed backward asymptotic tilings and two pairs of fixed forward asymptotic tilings for both φ and ψ. We solve the equations for the branch locus points in a manner similar to the foregoing examples:

$Br(\varphi) \pmod{\mathbb{Z}^2}$	$Br(\psi) \pmod{\mathbb{Z}^2}$
$(M-I)^{-1}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} \\ \frac{3}{4} \end{pmatrix}$	$(M-I)^{-1}\begin{pmatrix} -2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$
$(M-I)^{-1}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$	$(M-I)^{-1}\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} \\ \frac{3}{4} \end{pmatrix}$
$(M-I)^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}$	$(M-I)^{-1}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \end{pmatrix}$

There are infinitely many different triangles on the 2-torus having a given set of three points as vertices.

In case the vertices are the points of $Br(\varphi)$, we write the general edge vectors, up to integer translation in the lift, as $(\frac{1}{4} + p, \frac{1}{2} + q)$ and $(\frac{1}{4} + r, \frac{1}{4} + s)$. The area has the form

$$\left| \frac{1}{16} - \frac{1}{8} + \frac{1}{4}(p + s + 4ps - q - 2r - 4rq) \right|.$$

Since p, q, r, s are integers, the minimum area is $\frac{1}{16}$ (e.g. $p = q = r = s = 0$). The argument for $Br(\psi)$ is similar, but the minimum area is $\frac{1}{8}$. Were $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ homeomorphic, the map $\tau \circ T$ of Corollary 6 would preserve the areas of these triangles. We conclude they are not homeomorphic. $\square$

Example 5.

The pair

$$\varphi := \begin{cases} a \rightarrow aabbcac \\ b \rightarrow ababccbcc \\ c \rightarrow aaa^4 b^6 c^8 \end{cases} \qquad \psi := \begin{cases} a \rightarrow caabbac \\ b \rightarrow cababcbcc \\ c \rightarrow caaa^4 b^6 c^7 \end{cases}$$

provides an example of substitutions φ and ψ such that the Perron-Frobenius eigenvalue $\lambda_\varphi = \lambda_\psi$ is Pisot but not a unit. The reader can check that the branch locus $Br(\psi)$ is colinear, whereas $Br(\varphi)$ is not. By Theorem 1, $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ are not homeomorphic.   □

4. Proof of Theorem 1

Suppose that φ is a primitive and aperiodic substitution with *language*

$$Ł_\varphi := \{w \in \mathcal{A}^* : w \text{ is a factor of } \varphi^n(i) \text{ for some } i \in \mathcal{A}, n \in \mathbb{N}\}$$

and *allowed bi-infinite words*

$$X_\varphi := \{\cdots x_{-1} x_0 x_1 \cdots \mid x_n \in \mathcal{A}, x_n \cdots x_{n+k} \in Ł_\varphi \text{ for } n \in \mathbb{Z}, k \in \mathbb{N}\}.$$

Suppose that the finite subset $W = \{w_1, \ldots, w_k\} \subset Ł_\varphi$ has the properties that (1) each element of X_φ can be uniquely factored as a product of elements of W, and (2) for each $w_i \in W$, $\varphi(w_i)$ can be uniquely factored as a product of elements of W. We may then define a substitution $\varphi' : \{1, \ldots, k\} \rightarrow \{1, \ldots, k\}^*$ by the rule $\varphi'(i) = i_1 \cdots i_\ell$ provided $\varphi(w_i) = w_{i_1} \cdots w_{i_\ell}$. We will call such a φ' a *rewriting of* φ, and the morphism $\rho : \{1, \ldots, k\} \rightarrow \mathcal{A}^*$ given by $\rho(i) = w_i$ the associated *rewriting morphism*. Note that $\rho \circ \varphi' = \varphi \circ \rho$. It is not hard to see that φ' is also primitive and aperiodic.

Suppose now that φ' and φ are substitutions and ρ is a morphism with $\rho \circ \varphi' = \varphi \circ \rho$. Then the abelianizations A, A' and P of φ, φ' and ρ, resp., satisfy $PA' = AP$. Consequently, if w and w' are positive right Perron-Frobenius eigenvectors of A and A', resp., then $Pw' = aw$ for some $a > 0$, and A and A' have the same Perron-Frobenius eigenvalue. Thus φ' is Pisot if and only if φ is. In this case, let V and V' be the invariant Pisot subspaces (with rational bases) corresponding to A and A', resp. Since $w \in PV' \cap V$, $PV' \cap V$ is a nontrivial rational subspace of V. The characteristic polynomial of $A|_V$ is irreducible, and $PV' \cap V$ is A-invariant, which implies that $PV' \cap V = V$ and $V \subset PV'$. But $\dim V = d = \dim V'$, hence $P(V') = V$ and $P|_{V'} : V' \rightarrow V$ is invertible. Furthermore, if φ and φ' are Pisot, there is a natural continuous surjection $\overline{\rho} : \mathcal{T}_{\varphi'} \rightarrow \mathcal{T}_\varphi$ that conjugates Φ' with Φ, defined as follows: Given an edge $I = \sigma'_1 + x$ in V', with $\rho(i) = i_1 \cdots i_l$ and $v_j := \pi_V e_j$, let $\overline{\rho}(I)$ denote the finite strand in V defined by

$$\overline{\rho}(I) := \{\sigma_{i_1} + Px, \sigma_{i_2} + Px + v_{i_1}, \ldots, \sigma_{i_\ell} + Px + v_{i_1} + \cdots + v_{i_{\ell-1}}\}.$$

If $\gamma' = \{\sigma'_{i_k} + x_k : k \in \mathbb{Z}\}$ is any strand in V', let $\overline{\rho}(\gamma')$ denote the strand in V made up of the finite substrands $\overline{\rho}(\sigma_{i_k} + x_k)$ given by

$$\overline{\rho}(\gamma') = \bigcup_{k \in \mathbb{Z}} \overline{\rho}(\sigma'_{i_k} + x_k).$$

Observe that $\overline{\rho} \circ \Phi' = \Phi \circ \overline{\rho}$. Since $P\mathbb{E}^u_{\varphi'} = \mathbb{E}^u_\varphi$, there is an $R > 0$ such that $\overline{\rho}(\mathcal{T}_{\varphi'}) \subset \mathcal{F}^R_\varphi$. Thus if $\gamma' \in \mathcal{T}_{\varphi'}$,

$$\overline{\rho}(\gamma') = \overline{\rho}\left((\Phi')^n(\Phi')^{-n}(\gamma')\right) = \Phi^n(\overline{\rho}\left((\Phi')^{-n}(\gamma')\right) \in \Phi^n \mathcal{F}^R_\varphi$$

for all $n \in \mathbb{N}$, so

$$\overline{\rho}(\gamma') \in \bigcap_{n \in \mathbb{N}} \Phi^n(\mathcal{F}^R_\varphi) = \mathcal{T}_\varphi.$$

Continuity of $\overline{\rho}$ is clear, and from the fact that $\overline{\rho}(\gamma' - t) = \overline{\rho}(\gamma') - at$ and minimality of the tiling flow, it follows that $\overline{\rho} : \mathcal{T}_{\varphi'} \to \mathcal{T}_\varphi$ is surjective.

Suppose that V and V' are finite dimensional real vector spaces and $\Sigma \subset V$, $\Sigma' \subset V'$ are lattices. By a map $T : (V, \Sigma) \to (V', \Sigma')$, we will mean a linear transformation $T : V \to V'$ with $T(\Sigma) \subset \Sigma'$. Two maps $A : (V, \Sigma) \to (V, \Sigma)$ and $A' : (V', \Sigma') \to (V', \Sigma')$ are *shift-equivalent*, $A \sim_{SE} A'$, provided there are maps $T : (V, \Sigma) \to (V', \Sigma')$ and $S : (V', \Sigma') \to (V, \Sigma)$ and natural numbers m, n so that $A^m = ST$, $(A')^n = TS$, $A'T = TA$, and $SA' = AS$; S and T *express the shift equivalence*. The relation $\sim_{SE}$ is an equivalence relation, and if $A \sim_{SE} A'$, then $A^k \sim_{SE} (A')^k$ for all $k \in \mathbb{N}$. Furthermore, if $\hat{F}_T : \varprojlim F_A \to \varprojlim F_{A'}$ is defined by

$$\hat{F}_T(x_1 + \Sigma, x_2 + \Sigma, \ldots) = (Tx_1 + \Sigma', Tx_2 + \Sigma', \ldots)$$

(and $\hat{F}_S, \hat{F}_{A'}$ and $\hat{F}_A$ are defined similarly), and if Σ and Σ' are co-compact, then $\hat{F}_T$ and $\hat{F}_S$ are topological isomorphisms of the solenoids that conjugate the shifts $\hat{F}_A$ and $\hat{F}_{A'}$.

Lemma 3. *Suppose that φ and φ' are Pisot substitutions and that either*

(a) φ' is a rewriting of φ with rewriting morphism ρ, or

(b) there are morphisms α, β so that $\varphi = \alpha \circ \beta$ and $\varphi' = \beta \circ \alpha$.

Let A, V and Σ (A', V' and Σ', resp.) be the abelianization, invariant Pisot subspace and return lattice for φ (φ', resp.). Then, in the case of (a),

(1a) $\overline{\rho} : \mathcal{T}_{\varphi'} \to \mathcal{T}_\varphi$ is a homeomorphism,

or, in the case of (b),

(1b) $\overline{\alpha} : \mathcal{T}_{\varphi'} \to \mathcal{T}_\varphi$ and $\overline{\beta} : \mathcal{T}_\varphi \to \mathcal{T}_{\varphi'}$ are homeomorphisms.

Also,

(2) the maps $A|_V : (V, \Sigma) \to (V, \Sigma)$ and $A'|_{V'} : (V', \Sigma') \to (V', \Sigma')$ are shift equivalent.

Furthermore, if $g_\varphi : \mathcal{T}_\varphi \to \varprojlim F_{A|_V}$ and $g_{\varphi'} : \mathcal{T}_{\varphi'} \to \varprojlim F_{A'|_{V'}}$ are geometric realizations onto the natural solenoids, and $T : (V, \Sigma) \to (V', \Sigma')$ and $S : (V', \Sigma') \to (V, \Sigma)$ express the shift equivalence in (2), then there are translations τ and τ' on $\varprojlim F_{A|_V}$ and $\varprojlim F_{A'|_{V'}}$ so that, in the case of (a),

(3a) $g_\varphi \circ \overline{\rho} = \tau \circ \hat{F}_S \circ g_{\varphi'}$, and

$\qquad g_{\varphi'} \circ \overline{\rho}^{-1} = \tau' \circ \hat{F}_T \circ \hat{F}^m_{A|_V} \circ g_\varphi$ *for some $m \in \mathbb{N}$,*

or, in the case of (b),

(3b) $g_\varphi \circ \overline{\alpha} = \tau \circ \hat{F}_S \circ g_{\varphi'}$, and

$\qquad g_{\varphi'} \circ \overline{\beta} = \tau' \circ \hat{F}_T \circ g_\varphi.$

Proof: Suppose that φ' is a rewriting of φ with rewriting morphism ρ having abelianization P, and suppose that $\gamma', \gamma'' \in \mathcal{T}_{\varphi'}$ are such that $\overline{\rho}(\gamma') = \overline{\rho}(\gamma'')$. Let w' and w'' be the words of $X_{\varphi'}$ spelled out by γ' and γ'', where some location is chosen for the decimal point indicating the location of the 0th letter.

Then $\rho(w')$ and $\rho(w'')$ agree, up to shift, so that w' and w'' must agree, up to shift, by the uniqueness of factorization in X_φ in the definition of rewriting. It follows that $\gamma' = \gamma'' - t$ for some t. But then $\overline{\rho}(\gamma'') = \overline{\rho}(\gamma') = \overline{\rho}(\gamma'' - t) = \overline{\rho}(\gamma'') - at$, hence $t = 0$ and $\gamma' = \gamma''$. That is, $\overline{\rho}$ is a homeomorphism. In case $\varphi = \alpha \circ \beta$ and $\varphi' = \beta \circ \alpha$, we have that $\overline{\alpha} \circ \overline{\beta} = \Phi$ and $\overline{\beta} \circ \overline{\alpha} = \Phi'$, so both $\overline{\alpha}$ and $\overline{\beta}$ must be homeomorphisms.

Again, suppose that φ' is a rewriting of φ. If $v \in \Theta'(i)$ is a return vector for φ', say $\gamma' \in \mathcal{T}_{\varphi'}$ has edges I and $I + v$ of type i, then $\overline{\rho}\gamma' \in \mathcal{T}_\varphi$ has edges J and $J + Pv$ of type i_1, where $\rho(i) = i_1 \cdots$. Thus $Pv \in \Theta(i_1)$ is a return vector for φ. It follows that $P\Sigma' \subset \Sigma$.

While it is not necessarily the case that $P\Sigma' = \Sigma$, we shall see that $A^m\Sigma \subset P\Sigma'$ for some $m \in \mathbb{N}$. To this end, suppose that I is an edge in V of type i, and $I, I + v$ are edges of $\gamma \in \mathcal{T}_\varphi$ with $v \neq 0$. Let $\gamma' := \overline{\rho}^{-1}(\gamma)$, and for each $n \in \mathbb{N}$, let δ^n and η^n be maximal substrands of $(\Phi')^n(\gamma')$ with the property that $\overline{\rho}(\delta^n) \subset \Phi^n(I)$ and $\overline{\rho}(\eta^n) \subset \Phi^n(I + v)$. If $x^n, y^n \in Ł_{\varphi'}$ are the words corresponding to δ^n, η^n, resp., then we have the factorizations $p^n\rho(x^n)s^n$ and $q^n\rho(y^n)r^n$ of $\varphi^n(i)$ in which the words p^n, s^n, r^n, $q^n \in Ł_\varphi$ are of bounded length. Putting a decimal point (to mark the position of the 0th coordinate) on the left of $\varphi^n(i)$, we may choose $n_k \to \infty$ and $m_k \in \mathbb{N}$, with m_k on the order of $\frac{n_k}{2}$, so that $\sigma^{m_k}(\varphi^{n_k}(i))$ converges to a bi-infinite word $w \in X_\varphi$ (here σ is the shift that takes $.abc\cdots$ to $a.bc\cdots$). Let i^k be the letter (in $\mathcal{A} = \mathcal{A}_\varphi$) immediately to the right of the decimal point in $\sigma^{m_k}(\varphi^{n_k}(i))$, and let $x_0^{n_k}$ and $y_0^{n_k}$ denote the letters (in $\mathcal{A}' = \mathcal{A}_{\varphi'}$) of x^{n_k} and y^{n_k}, resp., so that $\rho(x_0^{n_k})$ and $\rho(y_0^{n_k})$ contain (the identified occurrence of) i^k. Putting a decimal point immediately to the left of $x_0^{n_k}$ and $y_0^{n_k}$ in x^{n_k} and y^{n_k}, resp., we may choose a subsequence n_{k_ℓ} so that $x^{n_{k_\ell}}$ and $y^{n_{k_\ell}}$ converge to bi-infinite words, say x and y, resp., in $X_{\varphi'}$. Now $\rho(x)$ and $\rho(y)$ give factorings of $w \in X_\varphi$ by the words $w_i = \rho(j)$, for $j \in \mathcal{A}'$. By uniqueness, $x = y$. Thus for large ℓ, $x_0^{n_{k_\ell}} = y_0^{n_{k_\ell}}$. Pick such an ℓ, let $m = n_{k_\ell}$, $j = x_0^m = y_0^m$, and $J, J + v'$ be the edges of the substrands δ^m, η^m of $(\Phi')^m(\gamma')$ corresponding to x_0^m and y_0^m, resp. Then $P(v') = A^m v$. That is, given $v \in \Theta(i)$, a return vector for φ, there is $v' \in \Theta'(j)$, a return vector for φ', with $v = (A|_V)^{-m}P(v') = P(A'|_{V'})^{-m}(v')$.

Now suppose that $v_1, \ldots, v_d$ is a basis for Σ, the return lattice for φ. Each v_j is then an integer combination of elements of $\bigcup_{k \in \mathbb{Z}}(A|_V)^k\Theta(i)$. The preceding argument shows that there is an $m \in \mathbb{N}$ so that $A^m v_j$ is in $P\Sigma'$ for all $j = 1, \ldots, d$. That is, $A^m\Sigma \subset P\Sigma'$. Let $S := P|_{V'}$ and $T := (P|_{V'})^{-1}(A|_V)^m$ (recall that $P|_{V'}$ is invertible because φ and φ' are Pisot). We have $S : (V', \Sigma') \to (V, \Sigma)$, $T : (V, \Sigma) \to (V', \Sigma')$, $A|_V S = S A'|_{V'}$, $T A|_V = A'|_{V'} T$, $ST = (A|_V)^m$, and $TS = (A'|_{V'})^m$.

Still in the rewriting case, suppose that $g_\varphi : \mathcal{T}_\varphi \to \varprojlim F_{A|_V}$ and $g_{\varphi'} : \mathcal{T}_{\varphi'} \to \varprojlim F_{A'|_{V'}}$ are geometric realizations onto the natural solenoids defined by the selection of $\{u_i\}$ in Γ/Σ and $\{u_i'\}$ in Γ'/Σ', resp. (see the definition of geometric realization in §2). Let $\{w_{ij}\} \subset \Gamma/\Sigma$ and $\{w_{ij}'\} \subset \Gamma'/\Sigma'$ denote the corresponding transition vectors (so $w_{ij} = u_j - u_i$ and $w_{ij}' = u_j' - u_i'$, for all i, j). Let $\gamma' \in \mathcal{T}_{\varphi'}$ have edge I of type i. Then $g_{\varphi'}(\gamma')$ has 0^{th}coordinate

$$(g_{\varphi'}(\gamma'))_0 = (\min I + \Sigma') - u_i'$$

in V'/Σ'. The strand $\overline{\rho}(\gamma')$ in $\mathcal{T}_\varphi$ has an edge J of type i_1, where $\rho(i) = i_1 \cdots$, with $\min J = P \min I$. Thus

$$(g_\varphi(\overline{\rho}(\gamma')))_0 = ((P \min I) + \Sigma) - u_{i_1}.$$

We have

$$
\begin{aligned}
F_P((g_{\varphi'}(\gamma'))_0) &= ((P\min I) + \Sigma) - F_P u_i' \\
&= \left(g_\varphi(\overline{\rho}(\gamma'))\right)_0 + u_{i_1} - F_P u_i'.
\end{aligned}
$$

We claim that $u_{i_1} - F_P u_i'$ is independent of $i \in \mathcal{A}'$. Indeed,

$$
\begin{aligned}
(u_{i_1} - F_P u_i') - (u_{j_1} - F_P u_j') &= (u_{i_1} - u_{j_1}) - (F_P(u_i' - u_j')) \\
&= w_{j_1 i_1} - F_P(w_{ji}').
\end{aligned}
$$

But if $\gamma' \in \mathcal{T}_{\varphi'}$ has edge I of type i and edge J of type j, so that $(\min J - \min I) + \Sigma = w_{ji}'$, then $\overline{\rho}(\gamma')$ has corresponding edges of type i_1 and j_1 with initial vertices differing by $P(\min J) - P(\min I)$. That is, $w_{j_1 i_1} = F_P(w_{ji}')$. It follows that if τ_1 is translation by $F_P u_i' - u_{i_1}$, then

$$
(g_\varphi \circ \rho)_0 = \tau_1 \circ F_{P|_{V'}} \circ (g_{\varphi'})_0.
$$

Similarly, k^{th} coordinates satisfy

$$
(g_\varphi \circ \rho)_k = \tau_1 \circ F_{P|_{V'}} \circ (g_{\varphi'})_k.
$$

Note that

$$
\begin{aligned}
F_{A|_V}(u_{i_1} - F_{P|_{V'}} u_i') &= F_{P|_V} u_{i_1} - F_{P|_V} F_{A|_V} u_i' \\
&= u_{i_1} - F_{P|_V} u_i'
\end{aligned}
$$

from the normalization requirement on the $\{u_i\}, \{u_i'\}$, so translation by this same element in each coordinate in $\varprojlim F_{A|_V}$ defines a translation τ on $\varprojlim F_{A|_V}$. We have

$$
g_\varphi \circ \overline{\rho} = \tau \circ \hat{F}_{P|_{V'}} \circ g_{\varphi'}.
$$

Thus with $S = P|_{V'}$ and $T = (P|_{V'})^{-1} \circ (A|_V)^m$, we have conclusion (3a).

The proofs of (2), in case $\varphi = \alpha \circ \beta$ and $\varphi' = \beta \circ \alpha$, and (3b) are similar (although more straightforward). $\hspace{2em}\square$

With the notation of Lemma 3, and with C_φ and $C_{\varphi'}$ the collections of special tilings for φ and φ' resp., note that $\overline{\rho}, \overline{\alpha}$, and $\overline{\beta}$ determine bijections between C_φ and $C_{\varphi'}$, and $\hat{F}_{A|_V}$ maps $g_\varphi(C_\varphi)$ bijectively onto itself. This yields the following consequence of Lemma 3.

Corollary 4. *If φ and φ' are Pisot substitutions with one a rewriting of the other, or for which there are morphisms α, β with $\varphi = \alpha \circ \beta$ and $\varphi' = \beta \circ \alpha$, then $A|_V : (V, \Sigma) \to (V, \Sigma)$ and $A'|_{V'} : (V', \Sigma') \to (V', \Sigma')$ are shift equivalent, expressed by maps $S : (V', \Sigma') \to (V, \Sigma)$ and $T : (V, \Sigma) \to (V', \Sigma')$, with*

$$
\tau \circ F_T \circ \pi_0 \circ g_\varphi(C_\varphi) = \pi_0 \circ g_{\varphi'}(C_{\varphi'})
$$

and

$$
\tau' \circ F_S \circ \pi_0 \circ g_{\varphi'}(C_{\varphi'}) = \pi_0 \circ g_\varphi(C_\varphi).
$$

Let us say that the substitutions φ and ψ are in the same *rewriting class* if there are substitutions $\varphi_0 = \varphi$, φ_1, ..., $\varphi_n = \psi$ with the property that for each $i = 0, \ldots, n-1$, one of φ_i and φ_{i+1} is a rewriting of the other. The proof of the rigidity result in [BSw2] (see Theorem 7 in the next section) establishes the following:

Lemma 5. *Suppose that φ and ψ are substitutions such that $\mathcal{T}_\varphi$ is homeomorphic with $\mathcal{T}_\psi$. Then there are $m, n \in \mathbb{N}$ and substitutions φ' and ψ' such that (1) φ' and φ^m are in the same rewriting class, as are ψ' and ψ^n, and (2) there are morphisms α and β such that $\varphi' = \alpha \circ \beta$ and $\psi' = \beta \circ \alpha$.*

Proof: See [BSw2, Theorem 2.1]. □

To prove Theorem 1, let φ and ψ be Pisot of degree d, with geometric realizations g_φ and g_ψ of $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$. Fix isomorphisms $L_\varphi : (V_\varphi, \Sigma_\varphi) \to (\mathbb{R}^d, \mathbb{Z}^d)$ and $L_\psi : (V_\psi, \Sigma_\psi) \to (\mathbb{R}^d, \mathbb{Z}^d)$, and let $Br(\varphi) = F_{L_\varphi} \circ \pi_0(g_\varphi(C_\varphi))$ and $Br(\psi) = F_{L_\psi} \circ \pi_0(g_\psi(C_\psi))$ be the corresponding branch loci. Let m, n, φ' and ψ' be as in Lemma 5. As $\mathcal{T}_\varphi = \mathcal{T}_{\varphi^m}$ and $\mathcal{T}_\psi = \mathcal{T}_{\psi^n}$, we may take $g_{\varphi^m} = g_\varphi$, $g_{\psi^n} = g_\psi$, $L_{\varphi^m} = L_\varphi$ and $L_{\psi^n} = L_\psi$, so that $Br(\varphi^m) = Br(\varphi)$ and $Br(\psi^m) = Br(\psi)$. Apply Corollary 4 repeatedly to get translations η and $\hat{\eta}$ (this last determined by η and L_ψ), and a map $T_1 : (V_\varphi, \Sigma_\varphi) \to (V_\psi, \Sigma_\psi)$ for which there are $k, \ell \in \mathbb{N}$ and a map $S_1 : (V_\psi, \Sigma_\psi) \to (V_\varphi, \Sigma_\varphi)$ such that

$$S_1 T_1 = (A_\varphi^m|_{V_\varphi})^k, \quad T_1 S_1 = (A_\psi^n|_{V_\psi})^\ell$$

and

$$
\begin{aligned}
Br(\psi) &= F_{L_\psi}\left((g_{\psi^n}(C_{\psi^n}))_0 \right) \\
&= F_{L_\psi}(\eta \circ F_{T_1}\left(g_{\varphi^m}(C_{\varphi^m}) \right)_0) \\
&= F_{L_\psi}\left(\eta \circ F_{T_1}\left(L_\varphi^{-1}(Br(\varphi)) \right) \right) \\
&= \hat{\eta} \circ F_{L_\psi T_1 L_\varphi^{-1}}.
\end{aligned}
$$

If $M_\varphi = L_\varphi A_\varphi L_\varphi^{-1}$, $M_\psi = L_\psi A_\psi L_\psi^{-1}$, $T = L_\psi T_1 L_\varphi^{-1}$, and $S = L_\varphi S_1 L_\psi^{-1}$ (expressed as matrices in the standard basis on $\mathbb{R}^d$), $m_0 = m^k$ and $\tau_0 = \hat{\eta}$, then $Br(\psi) = \tau_0 \circ F_T(Br(\varphi))$ with $ST = (M_\varphi)^{m_0}$. Similarly, if $m_1 = n^\ell$, there is a translation τ_1 with $Br(\varphi) = \tau_1 \circ F_S((Br(\psi))$ and $TS = (M_\psi)^{m_1}$. □

5. The Pisot Part of the Augmented Dimension Group

In this section we give a cohomological interpretation of Theorem 1. For a Pisot substitution φ of degree d, let L and M be as in the definition of $Br(\varphi)$. The induced $F_M : H^1(\mathbb{T}^d; \mathbb{R}) \hookleftarrow$ is then a vector space isomorphism with simple eigenvalue $\lambda = \lambda_\varphi$ and a $(d-1)$-dimensional invariant subspace complementary to the eigenspace of λ. This $(d-1)$-dimensional space splits $H^1(\mathbb{T}^d; \mathbb{R})$ into two closed invariant half-spaces, one of which contains the cocycle dual to $(F_L)_*([\omega])$, where $[\omega]$ is the 1-cycle generated by the positive eigenvector ω of $A = A_\varphi$. This half-space is called the *positive cone* in $H^1(\mathbb{T}^d; \mathbb{R})$ and is denoted by $H^1(\mathbb{T}^d; \mathbb{R})^+$. The inclusion $\iota : \mathbb{Z} \to \mathbb{R}$ induces $\iota^* : H^1(\mathbb{T}^d) \to H^1(\mathbb{T}^d; \mathbb{R})$ (an unspecified coefficient ring is understood to be $\mathbb{Z}$), and we define the positive cone in $H^1(\mathbb{T}^d)$ to be $H^1(\mathbb{T}^d)^+ := (\iota^*)^{-1}(H^1(\mathbb{T}; \mathbb{R})^+)$. Likewise, the positive cone in $H^1(\mathbb{T}^d, Br(\varphi))$ is

$$H^1(\mathbb{T}^d, Br(\varphi))^+ := (j^*)^{-1}\left(H^1(\mathbb{T}^d)^+ \right),$$

$j^* : H^1(\mathbb{T}^d, Br(\varphi)) \to H^1(\mathbb{T}^d)$ the natural homomorphism.

We have

$$F_M^* : \left(H^1(\mathbb{T}^d, Br(\varphi)), (H^1(\mathbb{T}^d, Br(\varphi))^+\right) \to \left(H^1(\mathbb{T}^d, Br(\varphi)), (H^1(\mathbb{T}^d, Br(\varphi))^+\right),$$

and we define $PDG(\varphi) := \varprojlim \left(F_B^* : H^1\left(\mathbb{T}^d, Br(\varphi)\right) \to H^1\left(\mathbb{T}^d, Br(\varphi)\right)\right)$, with positive cone

$$PDG(\varphi)^+ := \left\{[(k, g)] \in PDG(\varphi) : g \in H^1\left(\mathbb{T}^d, Br(\varphi)\right)^+\right\}$$

The *shift isomorphism* on $(PDG(\varphi), PDG(\varphi)^+)$ is the ordered isomorphism given by $[(k, g)] \mapsto \left[\left(k, F_M^*(g)\right)\right]$.
A homeomorphism of tiling spaces, $f : \mathcal{T}_\varphi \to \mathcal{T}_\psi$, is *orientation preserving (reversing)* provided it
takes the positive flow direction in $\mathcal{T}_\varphi$ to the positive (negative) flow direction in $\mathcal{T}_\psi$; that is, the function
$s : \mathcal{T}_\varphi \times \mathbb{R}^+ \to \mathbb{R}$ defined by $f(\gamma - t) = f(\gamma) - s(\gamma, t)$ is positive (negative).

Theorem 6. *If φ and ψ are Pisot substitutions, and the tiling spaces $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ are orientation preserving (reversing) homeomorphic, then there is an ordered isomorphism between $(PDG(\varphi), PDG(\varphi)^+)$ and $(PDG(\psi), PDG(\psi)^+)$ (resp. $(PDG(\psi), -PDG(\psi)^+)$) that conjugates some positive powers of the shift isomorphisms.*

Proof: Let T, S, τ_0, τ_1, m_0, and m_1 be as in Theorem 1. The commuting diagram

induces an isomorphism $[(k, g))] \to [(k, (\tau_1 \circ F_S)^*(g)]$ that conjugates the m_1^{th} power of the shift on
$PDG(\psi)$ with the m_0^{th} power of the shift on $PDG(\varphi)$. Since the order is dynamically defined, this
conjugacy alone guarantees that order is either preserved or reversed. If $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ are orientation
preserving homeomorphic, then S and T take $L_\psi(\omega_\psi)$ and $L_\varphi(\omega_\varphi)$ to positive multiples of $L_\varphi(\omega_\varphi)$ and
$L_\psi(\omega_\psi)$, resp. (here, L_φ and L_ψ denote the L in the definition of $Br(\varphi)$ and $Br(\psi)$, and ω_φ, ω_ψ are positive
eigenvectors of A_φ, A_ψ, resp.). Thus $(\tau_1 \circ F_S)^* \left(H^1(\mathbb{T}^d, Br(\varphi))^+\right) = H^1(\mathbb{T}^d, Br(\psi))^+$.
If $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ are orientation reversing homeomorphic, then $\mathcal{T}_\varphi$ and $\mathcal{T}_{\overline{\psi}}$ are orientation preserving home-
omorphic, $\overline{\psi}$ being the reverse of ψ (if $\psi(i) = i_1 \cdots i_k$ then $\overline{\psi}(i) = i_k \cdots i_1$). The above applied to
$(PDG(\varphi), PDG(\varphi)^+)$ and $(PDG(\psi), PDG(\psi)^+)$ together with the isomorphism $(PDG(\overline{\psi}), PDG(\overline{\psi})^+) \to$
$(PDG(\psi), -PDG(\psi)^+)$ induced by $F_{-\text{Id}}^*$ yields the conclusion of the theorem in the orientation reversing
case. $\square$

Example 6.

Consider $\varphi : \begin{cases} a & \to & ababba \\ b & \to & aabbba \end{cases}$ and $\psi : \begin{cases} a & \to & a^{47}ab^{18}bb^{29} \\ b & \to & a^{47}bb^{18}ab^{29} \end{cases}$.

One computes:

$$(PDG(\varphi), PDG(\varphi)^+) \cong \left(DG\begin{pmatrix} 6 & 0 \\ 1 & 1 \end{pmatrix}, DG\begin{pmatrix} 6 & 0 \\ 1 & 1 \end{pmatrix}^+ \right), \quad \text{and}$$

$$(PDG(\psi), PDG(\psi)^+) \cong \left(DG\begin{pmatrix} 96 & 0 \\ 19 & 1 \end{pmatrix}, DG\begin{pmatrix} 96 & 0 \\ 19 & 1 \end{pmatrix}^+ \right).$$

These groups are order isomorphic but, because no power of 96 is a power of 6, the shift isomorphisms do not have conjugate powers. By Theorem 6, $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ are not homeomorphic. $\qquad\square$

We now relate the pair $(PDG(\varphi), PDG(\varphi)^+)$ to the *ordered augmented cohomology* of the tiling space $\mathcal{T}_\varphi$. We begin with a description of the *augmented tiling space* (for more detail, and a precise description as an inverse limit, see [BSm]).

Let φ be any primitive aperiodic substitution with tiling space $\mathcal{T}_\varphi$. Let $\{\gamma_1^f, \ldots, \gamma_{n_f}^f\}$ be a collection of forward asymptotic special tilings, exactly one chosen from each forward asymptotic equivalence class, and $\{\gamma_1^b, \ldots, \gamma_{n_b}^b\}$ a collection of backward asymptotic special tilings, one from each backward asymptotic equivalence class. Let R_j^f, $j = 1, \ldots, n_f$, and R_i^b, $i = 1, \ldots n_b$, be the rays

$$R_j^f := \left\{ \{j\} \times \{\gamma_j^f - t\} \; : \; t \geq 0 \right\} \quad \text{and} \quad R_i^b := \left\{ \{i\} \times \{\gamma_i^b - t\} \; : \; t \geq 0 \right\}.$$

The augmented tiling space $\widetilde{\mathcal{T}}_\varphi$ is defined to be the union

$$\widetilde{\mathcal{T}}_\varphi := \mathcal{T}_\varphi \; \cup \; \left. \left(\bigcup_{j=1}^{n_f} R_j^f \cup \bigcup_{i=1}^{n_b} R_i^b \right) \right/ \left\{ \{j\} \times \{\gamma_j^f\}, \; \{i\} \times \{\gamma_i^b\} \right\},$$

in which all of the endpoints of the rays have been identified to a single branch point. The metric on $\mathcal{T}_\varphi$ is extended to $\widetilde{\mathcal{T}}_\varphi$ in such a way that

$$d\left(\gamma_j^f - t, \{j\} \times \{\gamma_j^f - t\} \right) = \frac{1}{1+t} = d\left(\gamma_i^b + t, \{i\} \times \{\gamma_i^b + t\} \right),$$

for $t \geq 0$, making the ray R_j^f asymptotic to the forward orbit of γ_j^f and the ray R_i^b asymptotic to the backward orbit of γ_i^b.

The homeomorphism Φ extends to $\widetilde{\Phi} : \widetilde{\mathcal{T}}_\varphi \hookleftarrow$ with $\widetilde{\Phi}\left(\{j\} \times \{\gamma_j^f - t\} \right) = \{j'\} \times \{\gamma_{j'}^f - \lambda t\}$, where j' is such that $\Phi(\gamma_j^f)$ is forward asymptotic to $\gamma_{j'}^f$, and similarly for the rays R_i^b. We order $H^1(\widetilde{\mathcal{T}}_\varphi)$ in such a way that $\widetilde{\Phi}^* : \left(H^1(\widetilde{\mathcal{T}}_\varphi), H^1(\widetilde{\mathcal{T}}_\varphi)^+ \right) \hookleftarrow$ is an order isomorphism. First note that $\Phi^* : H^1(\mathcal{T}_\varphi; \mathbb{R}) \hookleftarrow$ is a vector space isomorphism with simple eigenvalue $\lambda = \lambda_\varphi$ and codimension one invariant subspace complementary to the eigenspace of λ. This codimension one space splits $H^1(\mathcal{T}_\varphi)$ into two closed half-spaces, one of which corresponds to the positive direction of the flow[4]; call that space $H^1(\mathcal{T}_\varphi; \mathbb{R})^+$. Let $H^1(\widetilde{\mathcal{T}}_\varphi)^+ = (\iota_* j^*)^{-1} \left(H^1(\mathcal{T}_\varphi; \mathbb{R})^+ \right)$, where $\iota_* : H^1(\mathcal{T}_\varphi) \to H^1(\mathcal{T}_\varphi; \mathbb{R})$ is induced by $\iota : \mathbb{Z} \to \mathbb{R}$ and

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[4]Specifically, there is an orientation preserving map of $\mathcal{T}_\varphi$ onto the circle $\mathbb{T}^1$. Let $H^1(\mathcal{T}_\varphi; \mathbb{R})^+$ contain the positive half-line in $H^1(\mathbb{T}^1; \mathbb{R})$ pulled back to $H^1(\mathcal{T}_\varphi; \mathbb{R})$.

$j^* : H^1(\widetilde{\mathcal{T}}_\varphi) \to H^1(\mathcal{T}_\varphi)$ is induced by the inclusion $j : \mathcal{T}_\varphi \to \widetilde{\mathcal{T}}_\varphi$. The pair $\left(H^1(\widetilde{\mathcal{T}}_\varphi), H^1(\widetilde{\mathcal{T}}_\varphi)^+\right)$ is the *ordered augmented cohomology* of $\mathcal{T}_\varphi$. Call the isomorphism $\widetilde{\Phi}^* : \left(H^1(\widetilde{\mathcal{T}}_\varphi), H^1(\widetilde{T}_\varphi)^+\right) \hookleftarrow$ the **shift** isomorphism.

The rigidity result of [BSw2] is

Theorem 7. *If $f : \mathcal{T}_\varphi \to \mathcal{T}_\psi$ is a homeomorphism of substitution tiling spaces, then there are $m, n \in \mathbb{N}$ and $h : \mathcal{T}_\varphi \to \mathcal{T}_\psi$ isotopic to f so that $h \circ \Phi^m = \Psi^n \circ h$.*

Corollary 8. *If $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ are orientation preserving (reversing) homeomorphic, then the ordered cohomologies $\left(H^1(\widetilde{\mathcal{T}}_\varphi), H^1(\widetilde{\mathcal{T}}_\varphi)^+\right)$ and $\left(H^1(\widetilde{\mathcal{T}}_\psi), H^1(\widetilde{\mathcal{T}}_\psi)^+\right)$ (resp., $\left(H^1(\widetilde{\mathcal{T}}_\psi), -H^1(\widetilde{\mathcal{T}}_\psi)^+\right)$) are isomorphic by an isomorphism that conjugates some positive powers of the shift isomorphisms.*

Proof: The homeomorphism h of Theorem 7 extends to a homeomorphism $\widetilde{h} : \widetilde{\mathcal{T}}_\varphi \to \widetilde{\mathcal{T}}_\psi$ that conjugates $\widetilde{\Phi}^m$ with $\widetilde{\Psi}^n$. □

The corollary — without the conjugacy in the conclusion — appears in [BSm]. Also, in that paper, a nonnegative integer matrix $\widetilde{A}^t$ is constructed with ordered dimension group $\left(DG(\widetilde{A}^t), DG(\widetilde{A}^t)^+\right)$ isomorphic to $\left(H^1(\widetilde{\mathcal{T}}_\varphi), H^1(\widetilde{\mathcal{T}}_\varphi)^+\right)$.

Viewing $(PDG(\varphi), PDG(\varphi)^+)$ as the Pisot part of the augmented cohomology is justified by our final theorem. A formula for the Pisot part as the dimension group of an integer matrix will arise in the proof.

Theorem 9. *If φ is a Pisot substitution, then there is an ordered embedding of $(PDG(\varphi), PDG(\varphi)^+)$ into $\left(H^1(\widetilde{\mathcal{T}}_\varphi), H^1(\widetilde{\mathcal{T}}_\varphi)^+\right)$ that commutes with a positive power of the shift isomorphisms.*

Example 7.

If φ is the Thue-Morse substitution $(1 \to 12,\ 2 \to 21)$, then

$$(PDG(\varphi), PDG(\varphi)^+) = \left(\mathbb{Z}\left[\tfrac{1}{2}\right], \left\{x \in \mathbb{Z}\left[\tfrac{1}{2}\right],\ x \geq 0\right\}\right),$$

with shift isomorphism $x \mapsto 2x$, and

$$\left(H^1(\widetilde{\mathcal{T}}_\varphi), H^1(\widetilde{\mathcal{T}}_\varphi)^+\right) = \left(\mathbb{Z}\left[\tfrac{1}{2}\right] \oplus \mathbb{Z}^4, \left\{(x,y) \in \mathbb{Z}\left[\tfrac{1}{2}\right] \oplus \mathbb{Z}^4 :\ x \geq 0\right\}\right),$$

with shift isomorphism $(x, y) \mapsto (2x, y)$. □

Proof:(Theorem 9) Let φ be a Pisot substitution. To simplify, we pass to a *prepared* substitution (see [BSm]). We may suppose all special asymptotic tilings in $\mathcal{T}_\varphi$ are fixed by φ^n, for some $n \in \mathbb{N}$, and φ^n has a fixed bi-infinite word $\cdots b.a \cdots$. So we have $\varphi^n(a) = a \cdots$, $\varphi^n(b) = \cdots b$ and ba occurs in $\varphi^k(a)$, for some $k \in \mathbb{N}$.

We rewrite φ^n with stopping rule b and starting rule a as follows. Let $\mathcal{U} = \{u_1, \ldots, u_q\}$ be the finite collection of words with the properties

(*i*) each u_i occurs as a factor of the infinite word $a\varphi^n(a)\varphi^{2n}(a) \cdots$;

(*ii*) each u_i has the form $a \cdots b$.

(*iii*) ba is not a factor of any u_i, $i = 1, \ldots, q$.

Then each word $\varphi^n(u_i)$ factors uniquely in the form $u_{i_1} \cdots u_{i_p}$. From this, define ψ by $\psi(i) = i_1 \cdots i_p$, $i = 1, \ldots, q$. The substitution ψ is primitive and aperiodic Pisot ($\lambda_\psi = \lambda_\varphi^n$) and *strictly proper*: there are $r, s \in \{1, 2, \ldots, q\}$ so that $\psi(i) = r \cdots s$ for $i = 1, \ldots, q$. Also, the special asymptotic tilings in $\mathcal{T}_\psi$ are all fixed by Ψ. Then $\mathcal{T}_{\varphi^n} = \mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ are orientation preserving homeomorphic (this is a special case of Lemma 3) by a homeomorphism that conjugates Φ^n with Ψ. Hence, $\left(H^1(\widetilde{\mathcal{T}}_\varphi), H^1(\widetilde{\mathcal{T}}_\varphi)^+\right)$ and $\left(H^1(\widetilde{\mathcal{T}}_\psi), H^1(\widetilde{\mathcal{T}}_\psi)^+\right)$ are isomorphic by an isomorphism that conjugates $(\Phi^n)^*$ with Ψ^* (Corollary 8) and $(PDG(\varphi), PDG(\varphi)^+) = (PDG(\varphi^n), PDG(\varphi^n)^+)$ is isomorphic to $(PDG(\psi), PDG(\psi)^+)$ (Theorem 6). The isomorphism conjugates the $(\ell n)^{\text{th}}$ power of the shift on $(PDG(\varphi), PDG(\varphi)^+)$ to the ℓth power of the shift on $(PDG(\psi), PDG(\psi)^+)$ for some $\ell \in \mathbb{N}$. Thus once we prove the theorem for such substitutions ψ, we have proved it for all Pisot substitutions.

Let n_f denote the number of equivalence classes of forward asymptotic special tilings in $\mathcal{T}_\psi$, and n_b the number of special backward classes. As in [BSm], we may select special asymptotic tilings γ_j^f, $j = 1, \ldots, n_f$, one from each forward class, and γ_i^b, $i = 1, \ldots, n_b$, one from each backward class, with the properties

(i) if J is an edge of γ_j^f (resp., I an edge of γ_i^b) that meets $\mathbb{E}^s$, then J (resp., I) meets $\mathbb{E}^s$ in its interior;

(ii) if k is the type of J (resp., of I), then $\psi(k) = p_j^f k s_j^f$ (resp., $\psi(k) = p_i^b k s_i^b$) with p_j^f and s_j^f (resp., p_i^b and s_i^b) nonempty and

$$w_j^f := \cdots \psi^2(p_j^f) \psi(p_j^f) p_j^f \psi(s_j^f) \psi^2(s_j^f) \cdots \quad (\text{resp.}, \ w_i^b := \cdots \psi(p_i^b) p_i^b \psi(s_i^b) \cdots)$$

is the bi-infinite word corresponding to γ_j^f (resp., γ_i^b).

In particular, and this will be important later,

$$A(\min I) + \text{pr}_V [\![p_i^b]\!] = \min I \quad \text{and} \quad A(\max J) - \text{pr}_V [\![p_j^f]\!] = \max J,$$

where $[\![u]\!]$ denotes the abelianization of the word u.

Let E denote the $n \times (n_f + n_b - 1)$ matrix with ij^{th} entry

$$E_{ij} = \begin{cases} \text{number of occurrences of } i \text{ in } p_1^b s_j^f, & \text{if } 1 \leq j \leq n_f \\[2mm] \text{number of occurrences of } i \text{ in } p_{j-n_f+1}^b s_1^f, & \text{if } n_f < j \leq n_f + n_b - 1. \end{cases}$$

Let $A = A_\psi$ be the abelianization of ψ, and let I be the $(n_f + n_b - 1) \times (n_f + n_b - 1)$ identity matrix. The *augmented matrix* for ψ is

$$\widetilde{A} = \begin{pmatrix} A & E \\ O & I \end{pmatrix},$$

of size $\widetilde{n} = n + n_f + n_b - 1$. The *augmented dimension group* for ψ is the pair

$$\left(DG(\widetilde{A}^t), DG(\widetilde{A}^t)^+\right), \quad \text{where} \quad DG(\widetilde{A}^t) := \varinjlim \widetilde{A}^t : \mathbb{Z}^{\widetilde{n}} \hookleftarrow$$

with $DG(\widetilde{A}^t)^+$ determined (dynamiclly) as before:

Since λ_ψ is simple, $\widetilde{A}^t$ has a codimension one invariant subspace W (in $\mathbb{R}^{\widetilde{n}}$) complementary to the eigenspace of λ_ψ. Let

$$(\mathbb{Z}^{\widetilde{n}})^+ := \{x \in \mathbb{Z}^{\widetilde{n}} \ : \ x \text{ is in the half-space determined by } W \text{ containing}$$
$$\text{the nonnegative eigenvector corresponding to } \lambda_\psi\}.$$

Then $DG(\widetilde{A}^t)^+ := \left\{[k, g] \in DG(\widetilde{A}^t) \ : \ g \in (\mathbb{Z}^{\widetilde{n}})^+\right\}$.

We know from [BSm] that $\left(DG(\widetilde{A}^t), DG(\widetilde{A}^t)^+\right)$ is isomorphic to $\left(H^1(\tilde{T}_\psi), H^1(\tilde{T}_\psi)^+\right)$ via an isomorphism that conjugates the shifts. Our remaining task, then, is to embed $(PDG(\psi), PDG(\psi)^+)$ into $\left(DG(\widetilde{A}^t), DG(\widetilde{A}^t)^+\right)$.

Besides enabling us to describe its augmented cohomology as a dimension group, another advantage of a prepared substitution (like ψ) is that its return lattice Σ is the same as $\Gamma = \mathrm{pr}_V(\mathbb{Z}^n)$, making formulas for geometric realization simpler. To show these lattices agree, let $\gamma \in \mathcal{T}_\psi$, and let J be an edge of type j in γ followed by the edge I of type i. Since ψ is strictly proper, $\Psi(J)$ begins with an edge of the same type, say r, as does $\Psi(I)$. Thus the vector

$$\min \Psi(I) - \min \Psi(J) = \max \Psi(J) - \min \Psi(J) = \mathrm{pr}_V(Ae_j)$$
$$= (A|_V)\mathrm{pr}_V e_j = (A|_V)v_j \in \Theta(j).$$

So, $v_j \in \Sigma$ for all j, and $\Sigma = \Gamma$.

We may suppose, without loss of generality, that $\{v_1, \ldots, v_d\}$ is a basis for $\Sigma = \Gamma$, taken from the basis $\{v_1, \ldots, v_n\}$ for V. Let $\overline{Br}(\psi) = \pi_0 g_\psi(\mathcal{C}_\psi)$ be the branch locus in V/Σ (so $F_{L_\psi}\left(\overline{Br}(\psi)\right) = Br(\psi)$). If γ and γ' are asymptotic in $\mathcal{T}_\psi$, then $\pi_0 g_\psi(\gamma)$ and $\pi_0 g_\psi(\gamma')$ are asymptotic in V/Σ under the Kronecker flow; that is, $\pi_0 g_\psi(\gamma) = \pi_0 g_\psi(\gamma')$. It follows that

$$\overline{Br}(\psi) = \left\{\pi_0 g_\psi(\gamma_1^f), \ldots, \pi_0 g_\psi(\gamma_{n_f}^f)\right\} \bigcup \left\{\pi_0 g_\psi(\gamma_1^b), \ldots, \pi_0 g_\psi(\gamma_{n_b}^b)\right\}.$$

If I is the unique edge of γ_i^b that meets $\mathbb{E}^s$, let $x_i^b := \min I$; and if J is the unique edge of γ_j^f that meets $\mathbb{E}^s$, let $x_j^f := \max J$.

Since $\Sigma = \Gamma$, we may take the $u_i = 0$ in the definition of geometric realization g_ψ. Thus $\pi_0 g_\psi(\gamma_j^f) = x_j^f + \Sigma =: \overline{x}_j^f$ and $\pi_0 g_\psi(\gamma_i^b) = x_i^b + \Sigma =: \overline{x}_i^b$.

If $\#(\overline{Br}(\psi)) = 1$, then

$$(PDG(\psi), \ PDG(\psi)^+) \cong \left(\varinjlim A|_V, \ (\varinjlim A|_V)^+\right),$$

which is easily embedded in $(DG(\psi), (DG(\psi)^+)$.

Thus we can assume $\#Br(\psi) = m_f + m_b$ with $m_f > 0$ and $m_b > 0$, and after reindexing,

$$\overline{Br}(\psi) = \left\{\overline{x}_i^b \ : \ i = 1, \ldots, m_b\right\} \bigcup \left\{\overline{x}_j^f \ : \ j = 1, \ldots, m_f\right\}.$$

Recall that $\sigma_i := \{tv_i \ : \ t \in [0, 1]\}$ is the oriented segment. The homology classes $[\overline{\sigma}_i]$ of the oriented cycles $\overline{\sigma}_i := \sigma_i + \Sigma$, $i = 1, \ldots, d$, constitute a basis for $H_1(V/\Sigma)$.

For each $i = 1, \ldots, n_b$, let α_i denote the oriented line segment in V from x_i^b to 0; for each $j = 1, \ldots, n_f$, let β_j be the directed line segment from 0 to x_j^f; and, let $\overline{\alpha}_i := \alpha_i + \Sigma$, $\overline{\beta}_j := \beta_j + \Sigma$. Then

$$\left\{[\overline{\sigma}_1], \ldots, [\overline{\sigma}_\alpha], [\overline{\alpha}_1\overline{\beta}_1], [\overline{\alpha}_1\overline{\beta}_2], \ldots, [\overline{\alpha}_1\overline{\beta}_{m_f}], [\overline{\alpha}_2\overline{\beta}_1], [\overline{\alpha}_3\overline{\beta}_1], \ldots, [\overline{\alpha}_{m_b}\overline{\beta}_1]\right\}$$

is a basis for $H_1\left(V/\Sigma, \overline{Br}(\psi)\right)$.

Let $Ł : \mathbb{Z}^{\tilde{n}} \to \mathbb{Z}^{\tilde{n}}$ (recall $\tilde{n} = n + n_f + n_b - 1$) denote the homomorphism represented by $\tilde{A}$ in the standard basis $\mathcal{B} = \{e_1, \ldots, e_{\tilde{n}}\}$. For convenience, relabel the basis elements as follows:

$$e_{ij} = \begin{cases} e_i & \text{for } j = 0 \text{ and } i = 1, \ldots, n \\ e_{n+j} & \text{for } i = 1 \text{ and } j = 1, \ldots, n_f \\ e_{n+n_f+i-1} & \text{for } j = 2 \text{ and } i = 2, \ldots, n_b. \end{cases}$$

We define a homomorphism $P : \mathbb{Z}^{\tilde{n}} \to H_1\left(V/\Sigma, \overline{B}r(\psi)\right)$ on the basis $\mathcal{B}$ as follows:

$$P(e_{i0}) := \sum_{j=1}^d r_{ij}[\overline{\sigma}_j], \text{ provided } v_i = \sum_{j=1}^d r_{ij}v_j, \ i = 1, \ldots n;$$

$$P(e_{ij}) := [\overline{\alpha}_i\overline{\beta}_j], \ i = 1, \ j = 2, \ldots, n_f \text{ and } j = 1, \ i = 2, \ldots, n_b.$$

Clearly, P is surjective. The relation $PŁ = (F_{A|_V})_*P$ evidently holds on the basis elements $e_{10}, \ldots, e_{n0}$, so consider $e_{ij} \in \mathcal{B}$ with $j > 0$. Let $p_i := [\![p_i^b]\!] \in \mathbb{Z}^n$, $s_j = [\![s_j^f]\!] \in \mathbb{Z}^n$, and let $\iota : \mathbb{Z}^n \to \mathbb{Z}^{\tilde{n}}$ be given by $\iota(x_1, \ldots, x_n) = \sum_{i=1}^n x_i e_{i0}$. Then, $Ł(e_{ij}) = \iota(p_i) + \iota(s_j) + e_{ij}$ and

$$PŁ(e_{ij}) = PŁ(p_i) + PŁ(s_j) + P(e_{ij}) = \sum_{k=1}^d a_k[\overline{\sigma}_k] + \sum_{k=1}^d b_k[\overline{\sigma}_k] + [\overline{\alpha}_i\overline{\beta})j],$$

where $\text{pr}_V(p_i) = \sum_{k=1}^d a_k v_k$ and $\text{pr}_V(s_j) = \sum_{k=1}^d b_k v_k$.

On the other hand, $(F_{A|_V})_*P(e_{ij}) = (F_{A|_V})_*([\overline{\alpha}_i\overline{\beta}_j]) = [\overline{A(\alpha_i\beta_j)}]$, if $A(\alpha_i\beta_j)$ denotes the image of the directed curve $\alpha_i\beta_j$ under the linear map $A : V \to V$. We claim that $\overline{A(\alpha_i\beta_j)}$ is homologous to $\overline{\rho}_i + \overline{\alpha}_i\overline{\beta}_j + \overline{\eta}_j$, where $\rho_i := \{t\text{pr}_V(p_i) : 0 \le t \le 1\}$ and $\eta_j := \{t\text{pr}_V(s_j) : 0 \le t \le 1\}$ are directed segments.

Indeed, this follows from $A(-x_i^b) = \text{pr}_V(p_i) + (-x_i^b)$ and $Ax_j^f = x_j^f + \text{pr}_V(s_j)$ in the cover V of V/Σ (See Figure 5). Clearly $\overline{\rho}_i$ is homologous to $\sum_{k=1}^d a_k\overline{\sigma}_k$ and $\overline{\eta}_j$ is homologous to $\sigma_{k=1}^d b_k\overline{\sigma}_k$. Thus $PŁ(e_{ij}) = (F_{A|_V})_*P(e_{ij})$.

Since P is surjective, the dual

$$P^t : \text{Hom}\left(H_1\left(V/\Sigma, \overline{B}r(\psi)\right), \mathbb{Z}\right) \cong H^1\left(V/\Sigma, \overline{B}r(\psi)\right) \to \text{Hom}(\mathbb{Z}^{\tilde{n}}, \mathbb{Z}) \cong \mathbb{Z}^{\tilde{n}}$$

is injective. As $Ł^tP^t = P^t(F_{A|_V})^*$, we know P^t induces an injection

$$\widehat{P^t} : \varinjlim F_{A|V}^* = PDG(\psi) \to \varprojlim Ł^t = DG(\psi)$$

that commutes with the shifts. Since the positive cones are determined dynamically, $\widehat{P^t}$ either preserves or reverses order. As $[\sigma_1]^* \in H^1(V/\Sigma, \overline{B}r(\psi))^+$ and $\widehat{P^t}[\sigma_1]^* = [e_1]^* \in (DG(\psi))^+$, $\widehat{P^t}$ preserves order. Finally,
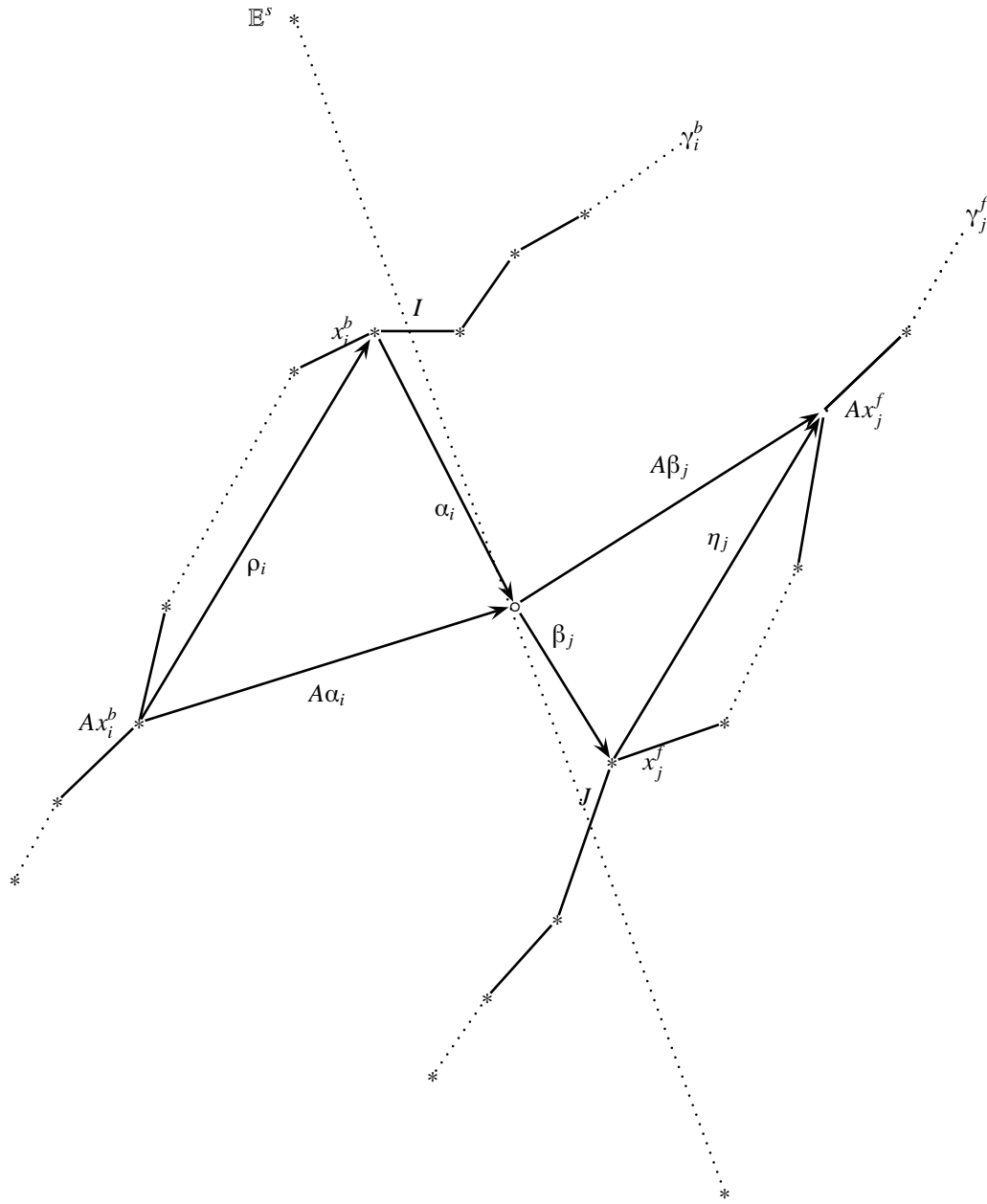
$$\widehat{F_{L_\psi}^*} : \varinjlim\left(F_{M_\psi}^* : H^1(\mathbb{T}^d, Br(\psi)) \hookleftarrow\right) = PDG(\psi) \to \varinjlim\left(F_{A|_V}^* : H^1\left(V/\Sigma, \overline{B}r(\psi)\right) \hookleftarrow\right)$$

by $\widehat{F_{L_\psi}^*}([(k, g)]) = \left[\left(k, F_{L_\psi}^*(g)\right)\right]$ is an ordered isomorphism (that commutes with the shifts), which embeds $(PDG(\psi), PDG(\psi)^+)$ into $(DG(\psi), DG(\psi)^+)$.

$\square$

FIGURE 1. A Homology

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